

A CHARACTERIZATION OF COMPLEX HYPERBOLIC KLEINIAN GROUPS IN DIMENSION 3 WITH TRACE FIELDS CONTAINED IN \mathbb{R}

JOONHYUNG KIM AND SUNGWOON KIM

ABSTRACT. We show that $\Gamma < \mathbf{SU}(3, 1)$ is a non-elementary complex hyperbolic Kleinian group in which $tr(\gamma) \in \mathbb{R}$ for all $\gamma \in \Gamma$ if and only if Γ is conjugate to a subgroup of $\mathbf{SO}(3, 1)$ or $\mathbf{SU}(1, 1) \times \mathbf{SU}(2)$.

1. INTRODUCTION

Let $\Gamma < \mathbf{SU}(2, 1)$ be a non-elementary complex hyperbolic Kleinian group. The *trace field* of Γ is the field generated by the traces of all the elements of Γ over the base field \mathbb{Q} . Maskit [5, Theorem V.G.18] characterized non-elementary hyperbolic Kleinian groups of $\mathbf{SL}(2, \mathbb{C})$ whose trace fields are contained in \mathbb{R} . The condition that the trace field of Γ is contained in \mathbb{R} is equivalent to that $tr(\gamma) \in \mathbb{R}$ for all $\gamma \in \Gamma$. In [1], X. Fu, L. Li and X. Wang showed that if $tr(\gamma) \in \mathbb{R}$ for all $\gamma \in \Gamma$, then Γ is Fuchsian. Here, a complex hyperbolic Kleinian group in dimension 2 is called *Fuchsian* if it keeps invariant a disc in Riemann sphere. It is very natural to generalize this result and there are two ways to generalize it, which are either Γ is a subgroup of $\mathbf{SU}(n, 1)$, where $n \geq 3$ or Γ is a subgroup of $\mathbf{Sp}(n, 1)$. In latter case, J. Kim proved in the case of $\mathbf{Sp}(2, 1)$ in [4].

In this paper, we consider the same problem in the case that Γ is a subgroup of $\mathbf{SU}(3, 1)$. Our main theorem is the following.

Theorem 1.1. *Let $\Gamma < \mathbf{SU}(3, 1)$ be a non-elementary complex hyperbolic Kleinian group. Then $tr(\gamma) \in \mathbb{R}$ for all $\gamma \in \Gamma$ if and only if Γ is conjugate to a subgroup of $\mathbf{SO}(3, 1)$ or $\mathbf{SU}(1, 1) \times \mathbf{SU}(2)$.*

The rest of this paper is organized as follows. In §2, we give some necessary preliminaries on complex hyperbolic spaces and in §3, we prove the main theorem.

¹2000 *Mathematics Subject Classification.* 22E40, 30F40, 57S30.

²*Key words and phrases.* Complex hyperbolic space, Complex hyperbolic Kleinian group, Cartan angular invariant.

³The first author was supported by NRF grant 2013-014376.

⁴The second author was partially supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (NRF-2012R1A1A2040663)

2. PRELIMINARIES

2.1. Complex hyperbolic space. Let $\mathbb{C}^{n,1}$ be a $(n+1)$ -complex vector space with a Hermitian form of signature $(n,1)$. An element of $\mathbb{C}^{n,1}$ is a column vector $z = (z_1, \dots, z_{n+1})^t$. Throughout this paper, we choose the second Hermitian form on $\mathbb{C}^{n,1}$ given by the matrix J

$$J = \begin{bmatrix} 0 & 0 & 1 \\ 0 & I_{n-1} & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Thus $\langle z, w \rangle = w^* J z = \bar{w}^t J z = z_1 \bar{w}_{n+1} + z_2 \bar{w}_2 + \dots + z_n \bar{w}_n + z_{n+1} \bar{w}_1$, where $z = (z_1, \dots, z_{n+1})^t$, $w = (w_1, \dots, w_{n+1})^t \in \mathbb{C}^{n,1}$.

Recall that the *Heisenberg group* is $\mathfrak{H} = \mathbb{C}^{n-1} \times \mathbb{R}$ with the group law

$$(z, u)(z', u') = (z + z', u + u' + 2\text{Im}\langle z, \bar{z}' \rangle),$$

where $\langle \cdot, \cdot \rangle$ is the standard Hermitian product on \mathbb{C}^{n-1} . One model of a complex hyperbolic space $\mathbf{H}_{\mathbb{C}}^n$, which matches the second Hermitian form is the **Siegel domain** \mathfrak{S} , which is parametrized in horospherical coordinates by $\mathfrak{H} \times \mathbb{R}_+$,

$$\psi : (z, u, v) \mapsto \begin{bmatrix} -\langle z, z \rangle - v + iu \\ \sqrt{2}z \\ 1 \end{bmatrix} \text{ for } (z, u, v) \in \overline{\mathfrak{S}} - \{\infty\} ; \psi : \infty \mapsto \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

where ∞ is a distinguished point at infinity. The boundary of \mathfrak{S} is given by $\mathfrak{H} \cup \{\infty\}$. Furthermore ψ maps \mathfrak{S} homeomorphically to the set of points w in $\mathbb{P}\mathbb{C}^{n,1}$ with $\langle w, w \rangle < 0$ and maps $\partial\mathfrak{S}$ homeomorphically to the set of points w in $\mathbb{P}\mathbb{C}^{n,1}$ with $\langle w, w \rangle = 0$.

There is a metric on \mathfrak{S} called the Bergman metric and the holomorphic isometry group of $\mathbf{H}_{\mathbb{C}}^n$ with respect to this metric is $\mathbf{PU}(n, 1)$. The elements of $\mathbf{PU}(n, 1)$ are classified by their fixed points. An element $A \in \mathbf{PU}(n, 1)$ is called *loxodromic* if it fixes exactly two points of $\partial\mathbf{H}_{\mathbb{C}}^n$, *parabolic* if it fixes exactly one point of $\partial\mathbf{H}_{\mathbb{C}}^n$, and called *elliptic* if it fixes at least one point of $\mathbf{H}_{\mathbb{C}}^n$.

Now let's consider $\mathbf{SU}(3, 1)$. A general form of an element $B \in \mathbf{SU}(3, 1)$ and its inverse are written as

$$B = \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ l & m & n & p \\ q & r & s & t \end{bmatrix}, \quad B^{-1} = \begin{bmatrix} \bar{t} & \bar{h} & \bar{p} & \bar{d} \\ \bar{r} & \bar{f} & \bar{m} & \bar{b} \\ \bar{s} & \bar{g} & \bar{n} & \bar{c} \\ \bar{q} & \bar{e} & \bar{l} & \bar{a} \end{bmatrix}.$$

Then, from $BB^{-1} = B^{-1}B = I$, we get the following identities.

$$\begin{aligned}
a\bar{t} + b\bar{r} + c\bar{s} + d\bar{q} &= 1, & a\bar{h} + b\bar{f} + c\bar{g} + d\bar{e} &= 0, & a\bar{p} + b\bar{m} + c\bar{n} + d\bar{l} &= 0, \\
a\bar{d} + |b|^2 + |c|^2 + d\bar{a} &= 0, & e\bar{t} + f\bar{r} + g\bar{s} + h\bar{q} &= 0, & e\bar{h} + |f|^2 + |g|^2 + h\bar{e} &= 1, \\
e\bar{p} + f\bar{m} + g\bar{n} + h\bar{l} &= 0, & l\bar{t} + m\bar{r} + n\bar{s} + p\bar{q} &= 0, & l\bar{p} + |m|^2 + |n|^2 + p\bar{l} &= 1, \\
q\bar{t} + |r|^2 + |s|^2 + t\bar{q} &= 0, & \bar{t}a + \bar{h}e + \bar{p}l + \bar{d}q &= 1, & \bar{t}b + \bar{h}f + \bar{p}m + \bar{d}r &= 0, \\
\bar{t}c + \bar{h}g + \bar{p}n + \bar{d}s &= 0, & \bar{t}d + |h|^2 + |p|^2 + \bar{d}t &= 0, & \bar{r}a + \bar{f}e + \bar{m}l + \bar{b}q &= 0, \\
\bar{r}b + |f|^2 + |m|^2 + \bar{b}r &= 1, & \bar{r}c + \bar{f}g + \bar{m}n + \bar{b}s &= 0, & \bar{s}a + \bar{g}e + \bar{n}l + \bar{c}q &= 0, \\
\bar{s}c + |g|^2 + |n|^2 + \bar{c}s &= 1, & \bar{q}a + |e|^2 + |l|^2 + \bar{a}q &= 0.
\end{aligned}$$

The following lemmas are needed for us.

Lemma 2.1 (Lemma 5.3 in [3]). *Let B in $\mathbf{SU}(3, 1)$ be such that the trace of B is real. Then the characteristic polynomial of B is self-dual.*

Lemma 2.2 (Proposition 2.2 in [4]). *For two nonzero complex numbers a and b , if ab and $a\bar{b}$ are all real, then either a and b are real or a and b are purely imaginary.*

Note that 0 is both a purely real and purely imaginary number.

2.2. Cartan angular invariant. The Cartan angular invariant is a well-known invariant in complex hyperbolic geometry, and here we give the definition and some properties which will be used in the proof of the main theorem. For more details, see [2].

The Cartan angular invariant $\mathbb{A}(x)$ of a triple $x = (x_1, x_2, x_3) \in (\partial\mathbf{H}_{\mathbb{C}}^n)^3$ is defined to be

$$\mathbb{A}(x) = \arg(-\langle \tilde{x}_1, \tilde{x}_2 \rangle \langle \tilde{x}_2, \tilde{x}_3 \rangle \langle \tilde{x}_3, \tilde{x}_1 \rangle),$$

where $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3$ are lifts of x_1, x_2, x_3 respectively. Then $\mathbb{A}(x)$ is independent of the choice of the lifts and $-\pi/2 \leq \mathbb{A}(x) \leq \pi/2$. Furthermore, $\mathbb{A}(x)$ is invariant under permutations of the points x_i up to sign.

Proposition 2.3. *A triple $x = (x_1, x_2, x_3) \in (\partial\mathbf{H}_{\mathbb{C}}^n)^3$ lies in the boundary of a complex line if and only if $\mathbb{A}(x) = \pm\pi/2$, and lies in the boundary of a Lagrangian plane if and only if $\mathbb{A}(x) = 0$.*

3. PROOF OF THE MAIN THEOREM

The “if” part is clear because any element of $\mathbf{SO}(3, 1)$ or $\mathbf{SU}(1, 1) \times \mathbf{SU}(2)$ has real trace, so we will prove the “only if” part.

It is well-known that a non-elementary Kleinian group contains infinitely many loxodromic elements (See [5] or [1]). Now let A be a loxodromic element fixing $\mathbf{0}$ and ∞ where $\mathbf{0}$ and ∞ denote the points of $\partial\mathbf{H}_{\mathbb{C}}^3$ represented by $(0, 0, 0, 1)$ and $(1, 0, 0, 0)$ respectively. In terms of matrices, due to the Lemma 2.1, we can write

$$A = \begin{bmatrix} u & 0 & 0 & 0 \\ 0 & e^{i\theta} & 0 & 0 \\ 0 & 0 & e^{-i\theta} & 0 \\ 0 & 0 & 0 & 1/u \end{bmatrix},$$

where $u > 1$. Up to conjugacy, we can assume that $A \in \Gamma$.

Lemma 3.1. *If $B = \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ l & m & n & p \\ q & r & s & t \end{bmatrix}$ is an arbitrary element of Γ , then a , t , and $f + n$ are real.*

Proof. Since the trace of every element in Γ is real, $\text{tr}(B)$ and $\text{tr}(AB) + \text{tr}(A^{-1}B)$ are real.

$$\begin{aligned} \text{tr}(B) &= a + t + f + n, \\ \text{tr}(AB) + \text{tr}(A^{-1}B) &= \left(u + \frac{1}{u}\right)(a + t) + 2\cos\theta(f + n). \end{aligned}$$

Solving for $(a + t)$ and $(f + n)$, since $u + \frac{1}{u} > 2\cos\theta$, we get that $a + t$ and $f + n$ are real. Now consider

$$\begin{aligned} \left(u - \frac{1}{u}\right)\text{tr}(B) + \text{tr}(AB) - \text{tr}(A^{-1}B) \\ = 2\left(u - \frac{1}{u}\right)a + \left(u - \frac{1}{u}\right)(f + n) + 2i(f - n)\sin\theta. \end{aligned}$$

Since $(f + n)$ is real, $\left(u - \frac{1}{u}\right)a + i(f - n)\sin\theta =: y_1 \in \mathbb{R}$.

Similarly, by considering

$$\begin{aligned} \left(u^2 - \frac{1}{u^2}\right)\text{tr}(B) + \text{tr}(A^2B) - \text{tr}(A^{-2}B) \\ = 2\left(u^2 - \frac{1}{u^2}\right)a + \left(u^2 - \frac{1}{u^2}\right)(f + n) + 2i(f - n)\sin 2\theta, \end{aligned}$$

we have $\left(u^2 - \frac{1}{u^2}\right)a + 2i(f - n)\sin\theta\cos\theta =: y_2 \in \mathbb{R}$. Hence,

$$\left(u + \frac{1}{u}\right)y_1 - y_2 = i(f - n)\sin\theta\left(u + \frac{1}{u} - 2\cos\theta\right) \in \mathbb{R}.$$

Since $u + \frac{1}{u} > 2\cos\theta$, $i(f - n)\sin\theta =: y_3$ is real, so $\left(u - \frac{1}{u}\right)a = y_1 - y_3$ is real and so a is real. Since $a + t$ is real, t is also real. \square

Lemma 3.2. *Consider the matrices A, B_1, B_2 in $\mathbf{SU}(3, 1)$.*

$$A = \begin{bmatrix} u & 0 & 0 & 0 \\ 0 & e^{i\theta} & 0 & 0 \\ 0 & 0 & e^{-i\theta} & 0 \\ 0 & 0 & 0 & 1/u \end{bmatrix}, B_1 = \begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ e_1 & f_1 & g_1 & h_1 \\ l_1 & m_1 & n_1 & p_1 \\ q_1 & r_1 & s_1 & t_1 \end{bmatrix}, B_2 = \begin{bmatrix} a_2 & b_2 & c_2 & d_2 \\ e_2 & f_2 & g_2 & h_2 \\ l_2 & m_2 & n_2 & p_2 \\ q_2 & r_2 & s_2 & t_2 \end{bmatrix},$$

where $u > 1$. Suppose that A, B_1 and B_2 are in Γ . Then $b_1e_2 + c_1l_2, d_1q_2, r_1h_2 + s_1p_2, q_1d_2, e_1b_2 + l_1c_2 + h_1r_2 + p_1s_2, f_1f_2 + g_1m_2 + m_1g_2 + n_1n_2$ are all real.

Proof. We already know that $a_1, a_2, t_1, t_2, f_1 + n_1, f_2 + n_2$ are real by Lemma 3.1. Since $(1, 1)$ entry of $B_1 B_2$ and $B_1 A B_2 + B_1 A^{-1} B_2$ are real, $a_1 a_2 + b_1 e_2 + c_1 l_2 + d_1 q_2$ and $\left(u + \frac{1}{u}\right) a_1 a_2 + 2 \cos \theta (b_1 e_2 + c_1 l_2) + \left(u + \frac{1}{u}\right) d_1 q_2$ are real, so $(b_1 e_2 + c_1 l_2) + d_1 q_2$ and $2 \cos \theta (b_1 e_2 + c_1 l_2) + \left(u + \frac{1}{u}\right) d_1 q_2$ are real. Solving for $d_1 q_2$ and $(b_1 e_2 + c_1 l_2)$, we get $d_1 q_2$ and $(b_1 e_2 + c_1 l_2)$ are real.

In a similar way, considering $(4, 4)$ entry of the same elements of Γ , we get that $q_1 d_2$ and $(r_1 h_2 + s_1 p_2)$ are real. Also, considering the sum of $(2, 2)$ entry and $(3, 3)$ entry of the same elements of Γ , we see that $e_1 b_2 + l_1 c_2 + h_1 r_2 + p_1 s_2, f_1 f_2 + g_1 m_2 + m_1 g_2 + n_1 n_2$ are all real. \square

Corollary 3.3. *Let B_1 and B_2 be arbitrary elements of Γ as written in Lemma 3.2.*

- (a) *Putting $B_1 = B_2$ in the lemma we see that $b_1 e_1 + c_1 l_1, d_1 q_1, r_1 h_1 + s_1 p_1$ and $f_1^2 + n_1^2 + 2m_1 g_1$ are all real.*
- (b) *Putting $B_2 = B_1^{-1}$ in the lemma we see that $b_1 \bar{r}_1 + c_1 \bar{s}_1$ and $d_1 \bar{q}_1$ are all real.*
- (c) *Either d_1 and q_1 are both real or else they are purely imaginary.*

Part (c) follows from (a), (b) and Lemma 2.2. By this corollary, we know that for any $B \in \Gamma$, either $(1, 4)$ entry and $(4, 1)$ entry of B are both real or else they are purely imaginary.

It is easy to check that $\mathbf{0}$ and ∞ are the fixed points of A . Since a non-elementary complex hyperbolic Kleinian group contains infinitely many loxodromic elements with pairwise distinct axes, there exists a loxodromic element B_0 of Γ such that the axes of A and B_0 are different. Write

$$B_0 = \begin{bmatrix} a_0 & b_0 & c_0 & d_0 \\ e_0 & f_0 & g_0 & h_0 \\ l_0 & m_0 & n_0 & p_0 \\ q_0 & r_0 & s_0 & t_0 \end{bmatrix}.$$

Then we claim that $d_0 q_0 \neq 0$. If $d_0 = 0$, then we get $h_0 = p_0 = 0$ from the identity $\bar{t}_0 d_0 + |h_0|^2 + |p_0|^2 + \bar{d}_0 t_0 = 0$. This implies B_0 fixes $\mathbf{0}$. Similarly if $q_0 = 0$, it can be easily seen that B_0 fixes ∞ . In other words, if $d_0 q_0 = 0$, then B_0 fixes either $\mathbf{0}$ or ∞ . This means that A and B_0 share one but both fixed points. However the subgroup generated by such A and B_0 is not discrete, which contradicts that Γ is discrete. Therefore the claim holds. Now we will consider the following two cases separately.

Case I: d_0 and q_0 are purely imaginary.

From the identity $a_0 \bar{d}_0 + |b_0|^2 + |c_0|^2 + d_0 \bar{a}_0 = 0$, we have $b_0 = c_0 = 0$ because $a_0 \bar{d}_0 + d_0 \bar{a}_0 = 0$. Similarly, from identities $\bar{q}_0 a_0 + |e_0|^2 + |l_0|^2 + \bar{a}_0 q_0 = 0$, $q_0 \bar{t}_0 + |r_0|^2 + |s_0|^2 + t_0 \bar{q}_0 = 0$, and $\bar{t}_0 d_0 + |h_0|^2 + |p_0|^2 + \bar{d}_0 t_0 = 0$, we get

$e_0 = l_0 = 0$, $r_0 = s_0 = 0$ and $h_0 = p_0 = 0$, respectively. Hence

$$B_0 = \begin{bmatrix} a_0 & 0 & 0 & d_0 \\ 0 & f_0 & g_0 & 0 \\ 0 & m_0 & n_0 & 0 \\ q_0 & 0 & 0 & t_0 \end{bmatrix},$$

where a_0, t_0 are real and d_0, q_0 are purely imaginary. Furthermore, since $\det B_0 = (a_0 t_0 - d_0 q_0)(f_0 n_0 - g_0 m_0) = 1$, we have $f_0 n_0 - g_0 m_0 = 1$ because $1 = a_0 \bar{t}_0 + b_0 \bar{r}_0 + c_0 \bar{s}_0 + d_0 \bar{q}_0 = a_0 t_0 - d_0 q_0$.

From $\bar{B}_0^t J B_0 = J$, we have

$$\begin{bmatrix} \bar{a}_0 & \bar{q}_0 \\ \bar{d}_0 & \bar{t}_0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_0 & d_0 \\ q_0 & t_0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} \bar{f}_0 & \bar{m}_0 \\ \bar{g}_0 & \bar{n}_0 \end{bmatrix} \begin{bmatrix} f_0 & g_0 \\ m_0 & n_0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

where $a_0 t_0 - d_0 q_0 = f_0 n_0 - g_0 m_0 = 1$. This implies that $\begin{bmatrix} a_0 & d_0 \\ q_0 & t_0 \end{bmatrix} \in \mathbf{SU}(1, 1)$

and $\begin{bmatrix} f_0 & g_0 \\ m_0 & n_0 \end{bmatrix} \in \mathbf{SU}(2)$. Hence B_0 is an element of $\mathbf{SU}(1, 1) \times \mathbf{SU}(2)$.

Now let $B = \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ l & m & n & p \\ q & r & s & t \end{bmatrix}$ be any other element of Γ . Then a and t are

real. By Lemma 3.2, dq_0 is real and so d is purely imaginary because q_0 is a non-zero purely imaginary number. From identities $a\bar{d} + |b|^2 + |c|^2 + d\bar{a} = 0$ and $\bar{t}d + |h|^2 + |p|^2 + d\bar{t} = 0$, we get $b = c = p = h = 0$. Similarly, since $d_0 q$ is real and d_0 is a non-zero purely imaginary number, we have that q is purely imaginary and using some identities, we get $e = l = r = s = 0$. Using the same arguments as above, we conclude that B is of the form

$$B = \begin{bmatrix} a & 0 & 0 & d \\ 0 & f & g & 0 \\ 0 & m & n & 0 \\ q & 0 & 0 & t \end{bmatrix},$$

where $at - dq = fn - gm = 1$. Thus we can conclude that Γ is a subgroup of $\mathbf{SU}(1, 1) \times \mathbf{SU}(2)$ defined by

$$\mathbf{SU}(1, 1) \times \mathbf{SU}(2) := \left\{ \begin{bmatrix} a & 0 & 0 & d \\ 0 & f & g & 0 \\ 0 & m & n & 0 \\ q & 0 & 0 & t \end{bmatrix} : \begin{bmatrix} a & d \\ q & t \end{bmatrix} \in \mathbf{SU}(1, 1), \begin{bmatrix} f & g \\ m & n \end{bmatrix} \in \mathbf{SU}(2) \right\}$$

Case II: d_0 and q_0 are real.

Let $B = \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ l & m & n & p \\ q & r & s & t \end{bmatrix}$ be any other element of Γ . Then, according to

Lemma 3.1, a and t are real. By Lemma 3.2, dq_0 and qd_0 are real. Since

d_0 and q_0 are non-zero real numbers, d and q are real. Hence we know that $(1, 1), (1, 4), (4, 1)$ and $(4, 4)$ entries of any element of Γ are real. Let B_1 and B_2 be elements of Γ as written in Lemma 3.2. Considering the $(1, 4)$ entry of $B_1^{-1}B_2$, we have that $\bar{t}_1d_2 + \bar{h}_1h_2 + \bar{p}_1p_2 + \bar{d}_1t_2$ is real. Noting that $B \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^t = \begin{bmatrix} d & h & p & t \end{bmatrix}^t$ and

$$\langle B_2 \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^t, B_1 \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^t \rangle = \bar{t}_1d_2 + \bar{h}_1h_2 + \bar{p}_1p_2 + \bar{d}_1t_2,$$

it follows that $\langle B_2 \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^t, B_1 \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^t \rangle$ is real for all $B_1, B_2 \in \Gamma$.

Let V be the \mathbb{R} -linear span of $\{B \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^t : B \in \Gamma\}$. Then it can be easily seen that V is totally real. Furthermore every element of Γ stabilizes V . Therefore Γ leaves a totally real subspace of $\mathbf{H}_{\mathbb{C}}^3$ invariant. This means that Γ is conjugate to a subgroup of $\mathbf{S}(\mathbf{O}(2, 1) \times \mathbf{O}(1))$ or $\mathbf{SO}(3, 1)$. Since $\mathbf{S}(\mathbf{O}(2, 1) \times \mathbf{O}(1))$ is a subgroup of $\mathbf{SO}(3, 1)$, we finally conclude that Γ is conjugate to a subgroup of $\mathbf{SO}(3, 1)$.

REFERENCES

- [1] X. Fu, L. Li, X. Wang, A characterization of Fuchsian groups acting on complex hyperbolic spaces. Czechoslovak Math. J. 62 (137) (2012), no. 2, 517–525.
- [2] W. M. Goldman, Complex hyperbolic Geometry, Oxford Univ. Press, (1999).
- [3] K. Gongopadhyay and J. R. Parker, Reversible complex hyperbolic isometries, Linear Algebra and its Applications 438 (2013), no. 6, 2728–2739.
- [4] J. Kim, Quaternionic hyperbolic Fuchsian groups, Linear Algebra and its Applications 438 (2013), no. 9, 3610–3617.
- [5] B. Maskit, Kleinian groups, Springer-Verlag, (1988).

JOONHYUNG KIM, DEPARTMENT OF MATHEMATICS, KONKUK UNIVERSITY, 1 HWAYANG-DONG, GWANGJIN-GU, SEOUL 143-701, REPUBLIC OF KOREA

E-mail address: calvary@snu.ac.kr

SUNGWOON KIM, CENTER FOR MATHEMATICAL CHALLENGES, KIAS, HOEGIRO 85, DONGDAEMUN-GU, SEOUL 130-722, REPUBLIC OF KOREA

E-mail address: sungwoon@kias.re.kr